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# ON GALILEI INVARIANCE IN QUANTUM MECHANICS AND THE BARGMANN SUPERSELECTION RULE

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## Abstract

We reinvestigate Bargmann's superselection rule for the overall mass of  $n$  particles in ordinary quantum mechanics with Galilei invariant interaction potential. We point out that in order for mass to define a superselection rule it should be considered as a dynamical variable. We present a minimal extension of the original dynamics in which mass is treated as dynamical variable. Here the classical symmetry group turns out to be given by an  $\mathbf{R}$ -extension of the Galilei group which formerly appeared only at the quantum level. There is now no obstruction to implement an action of the classical symmetry group on Hilbert space. We include some comments of a general nature on formal derivations of superselection rules without dynamical context.

## Introduction

It seems to be a generally accepted text-book wisdom that non-relativistic quantum mechanics has superselection rules for the total mass  $M$  [1-7]. This means that the superposition of two states,  $\psi_+ = \psi_M + \psi_{M'}$ , corresponding to different overall masses,  $M$  and  $M'$ , does not define a pure state. This is sometimes expressed by saying that such superpositions are forbidden. Formally this really means that the matrix elements  $\langle\psi_M|\mathcal{O}|\psi_{M'}\rangle$  are zero for all observables  $\mathcal{O}$ , which is equivalent to saying that the two density matrices  $\rho_+ = |\psi_+\rangle\langle\psi_+|$  and  $\rho_{\text{mix}} = |\psi_M\rangle\langle\psi_M| + |\psi_{M'}\rangle\langle\psi_{M'}|$  define the *same* expectation value functional on all observables, i.e.,

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$\text{tr}(\rho_+ \mathcal{O}) = \text{tr}(\rho_{\text{mix}} \mathcal{O})$  for all  $\mathcal{O}$ . In particular,  $\psi_+$  does not define a pure state on the observables.

At this point one must wonder how this statement, which is usually “derived” within standard quantum mechanics, should actually be interpreted within that framework. It obviously refers to a single system whose set of pure states contains  $\psi_M$  and  $\psi_{M'}$ . But precisely what is that system? In ordinary quantum mechanics, the masses are fixed parameters which do not label different states but rather belong to the specification of the system. In other words, two  $n$ -particle systems with different overall mass are really considered to be *different* systems. In order to regard  $\psi_M$  and  $\psi_{M'}$  as states of the *same* system, the label  $M$  must refer to some dynamical variable. Mass must therefore be treated dynamically and the quantum theory should contain a corresponding total mass operator  $M$ . That it defines a superselection rule is then equivalent to saying that  $M$  lies in the centre of the algebra of observables. But once the total mass becomes a dynamical variable, there is at least no a priori reason to restrict the observables to those commuting with  $M$ . We thus face the following situation: Standard a priori derivations within non-relativistic Schrödinger theory do not treat total mass as dynamical variable and hence lack a proper interpretation. It is true that many texts refer to some “mass operator” but, to our knowledge, a dynamical context is never specified. On the other hand, if mass is a dynamical variable, there is no a priori reason for a mass superselection rule. In order to derive it, one needs additional inputs which must be different in character from mere formal consistency conditions<sup>1</sup>. But such a derivation has not yet been given.

We stress that in principle the specification of a dynamical law is necessary to find the right implementation of the Galilei group (or an extension thereof) on state space, since it should be implemented as a (dynamical) symmetry. Specific properties of the implementation should therefore not be considered independent of the dynamical context. We will explain the details of the implementation in the next section which leads to the precise statement of Galilei invariance of standard quantum mechanics<sup>2</sup>. This is done not for a free particle [1-6] but in the more

<sup>1</sup> In field theory such an additional input is, for example, given by the principle of locality. There it is the restriction to (quasi-) local observables that causes the algebra of observables to acquire a non-trivial centre.

<sup>2</sup> Sometimes purely kinematical symmetry groups are invoked in the definition of quantum mechanical state spaces, before any dynamical laws are given. See e.g. [8]. This should be distinguished from the dynamical notion of symmetry used here.

general context of  $n$  spinless particles with Galilei-invariant potential. In particular we learn that it is not the Galilei group that acts on the Hilbert space but a central extension thereof. We then recall how this implies the standard argument for the existence of a superselection rule for overall mass. In section 2 we present a very simple, minimal generalization of the classical Hamiltonian system which includes the masses as dynamical variables. It turns out that here the central extension of the Galilei group – formerly only needed to implement the symmetry group in Hilbert space – now already appears at the classical level. In section 3 we discuss the Schrödinger equation of this extended model. There is now no discrepancy between the classical symmetry group and the symmetry group that acts on the Hilbert space, and hence no a priori reason from kinematics for a superselection rule. We end with a brief discussion section.

## Section 1

We denote the Galilei group by  $\mathcal{G}$  and parameterize it by an  $SO(3)$  rotation matrix  $R$ , a boost velocity vector  $\vec{v}$ , a space translation vector  $\vec{a}$ , and a real-valued time translation  $b$ . To avoid going into topological considerations and also to accommodate half-integer spin we should actually take  $SU(2)$  instead of  $SO(3)$ . In order to not complicate the notation we can do this implicitly by regarding  $R$  as an  $SU(2)$  element whose action on  $R^3$  vectors is via the  $SO(3)$  projection. A group element is thus denoted by  $g = (R, \vec{v}, \vec{a}, b)$  and the laws for multiplication and forming the inverse is given by

$$\begin{aligned} g'g &= (R', \vec{v}', \vec{a}', b')(R, \vec{v}, \vec{a}, b) \\ &= (R'R, \vec{v}' + R'\vec{v}, \vec{a}' + R'\vec{a} + \vec{v}'b, b' + b) \end{aligned} \tag{1.1}$$

$$g^{-1} = (R, \vec{v}, \vec{a}, b)^{-1} = (R^{-1}, -R^{-1}\vec{v}, -R^{-1}(\vec{a} - \vec{v}b), b). \tag{1.2}$$

We consider  $n$  point-particles of individual masses  $m_i$  interacting via a Galilei invariant potential  $V$ . The classical configuration space is  $R^{3n}$  coordinatized by the particle positions  $\{\vec{x}_i\}$ . On this configuration space  $\mathcal{G}$  acts in the standard way:

$$(\{\vec{x}_i\}, t) \xrightarrow{g} g(\{\vec{x}_i\}, t) = (\{R\vec{x}_i + \vec{a} + \vec{v}t\}, t + b). \tag{1.3}$$

$V$  is Galilei invariant, if and only if it only depends on the  $\frac{1}{2}n(n-1)$  distances  $r_{ij} := |\vec{x}_i - \vec{x}_j|$ . In particular it is time independent. The Schrödinger equation

reads

$$\mathsf{L}\psi := (i\hbar\partial_t - \mathsf{H}) \psi = 0 \quad (1.4)$$

$$\mathsf{H} = -\sum_{i=1}^n \frac{\hbar^2}{2m_i} \Delta_i + V(\{r_{ij}\}), \quad (1.5)$$

where we set  $\Delta_i = \vec{\nabla}_i \cdot \vec{\nabla}_i$  (no summation over  $i$ ). The Hilbert space is given by  $\mathcal{H} = L^2(R^{3n}, d^{3n}x)$ , with  $d^{3n}x$  the standard Lebesgue measure on  $R^{3n}$ . The Schrödinger equation is integrated by a one-parameter group of unitary evolution operators,  $\exp(-\frac{i}{\hbar}\mathsf{H}t)$ , which provide the following bijective correspondence between  $\mathcal{H}$  and the space of solutions  $\psi(\{\vec{x}_i\}, t)$  to the Schrödinger equation (1.4):

$$\psi(\{\vec{x}_i\}, t) \leftrightarrow \exp\left(\frac{i}{\hbar}\mathsf{H}t\right) \psi(\{\vec{x}_i\}, t) = \psi(\{\vec{x}_i\}, t=0) \in \mathcal{H}. \quad (1.6)$$

We now try to consider the Galilei group as symmetry group in the quantum theory. A priori it is not obvious how an element  $g \in \mathcal{G}$  should act on  $\mathcal{H}$ . It is important to note that we do not just wish to find any unitary action, of which there are clearly many, but rather the particular action that corresponds to a symmetry for the dynamical equation (1.4). The idea is to use the identification (1.6) and first determine  $g$ 's action on the solutions of  $\mathsf{L}\psi = 0$ . These are functions of the coordinates  $(\{\vec{x}_i\}, t)$  on which  $g$ 's action is given by (1.3). An obvious choice would therefore consist in shifting the function  $\psi$  along  $g$  by taking the composite function  $\psi \circ g^{-1}$ , just like for a classical field. But one may easily check that the shifted function does not solve the Schrödinger equation anymore. This can be remedied by also multiplying  $\psi$  with a phase-function  $\exp(if_g)$  which we now determine. We set

$$T : \psi \xrightarrow{g} \mathsf{T}_g \psi = \exp(if_g) (\psi \circ g^{-1}) \quad (1.7)$$

and impose the requirement that the resulting function again solves the Schrödinger equation. Acting with  $\mathsf{L}$  on  $\mathsf{T}_g \psi$  one finds

$$\begin{aligned} \mathsf{L}(\mathsf{T}_g \psi) &= \exp(if_g) (\mathsf{L}\psi \circ g^{-1}) \\ &\quad + i\hbar \exp(if_g) \sum_{i=1}^n \left( \frac{\hbar}{m_i} \vec{\nabla}_i f_g - \vec{v} \right) \cdot R \cdot \left( \vec{\nabla}_i \psi \circ g^{-1} \right) \\ &\quad - \hbar \exp(if_g) \left( \partial_t f_g + \sum_{i=1}^n \frac{\hbar}{2m_i} (\vec{\nabla}_i f_g)^2 - i \sum_{i=1}^n \frac{\hbar}{2m_i} \Delta_i f_g \right) (\psi \circ g^{-1}). \end{aligned} \quad (1.8)$$

This shows that  $T_g$  transforms solutions of the Schrödinger equation to solutions, if and only if the extra two terms in (1.8) vanish. This is equivalent to

$$f_g(\{\vec{x}_i\}, t) = \frac{M}{\hbar} \left( \vec{v} \cdot \vec{R} - \frac{1}{2} \vec{v}^2 t + c_g \right), \quad (1.9)$$

where  $M = \sum_i m_i$  is the total mass and  $\vec{R} = \frac{1}{M} \sum_i m_i \vec{x}_i$  the centre-of-mass-vector.  $c_g$  is a  $g$ -dependent integration constant. A convenient choice is  $c_g = \frac{1}{2} \vec{v}^2 b - \vec{v} \cdot \vec{a}$ , which leaves us with the transformation law

$$(T_g \psi)(\{\vec{x}_a\}, t) = \exp \left\{ \frac{i}{\hbar} M \left[ \vec{v} \cdot (\vec{R} - \vec{a}) - \frac{1}{2} \vec{v}^2 (t - b) \right] \right\} \psi(g^{-1}(\{\vec{x}_i\}, t)). \quad (1.10)$$

The fact that  $T_g \psi$  satisfies the Schrödinger equation<sup>3</sup> can be equivalently expressed by  $(T_g \psi)(t) = \exp(-\frac{i}{\hbar} \mathbf{H} t)((T_g \psi)(t=0))$ , which means that we just need to put  $t=0$  in (1.10) in order to obtain  $g$ 's action on  $\mathcal{H}$ , which we call  $U : g \rightarrow U_g$ ,

$$\begin{aligned} U_g \psi(\{\vec{x}_i\}) &= \exp \left\{ \frac{i}{\hbar} M \left[ (\vec{v} \cdot (\vec{R} - \vec{a}) + \frac{1}{2} \vec{v}^2 b) \right] \right\} \\ &\times (\exp(\frac{i}{\hbar} \mathbf{H} b) \psi)(\{R^{-1}(\vec{x}_i - \vec{a} + \vec{v} b)\}). \end{aligned} \quad (1.11)$$

This action is quite obviously unitary.

However, having found the transformation law for each  $g \in \mathcal{G}$  does not imply a representation of  $\mathcal{G}$  on  $\mathcal{H}$ . In fact one now finds a phase difference between the Galilei transformation  $U_{g'g}$  and the composite transformation  $U_{g'} U_g$ :

$$U_{g'} U_g = \exp \left( \frac{i}{\hbar} M \xi(g', g) \right) U_{g'g} \quad (1.12)$$

$$\xi(g', g) = \vec{v}' \cdot R' \cdot \vec{a} + \frac{1}{2} \vec{v}'^2 b. \quad (1.13)$$

It is straightforward to check the condition

$$\delta \xi(g'', g', g) := \xi(g''g', g) - \xi(g'', g'g) + \xi(g'', g') - \xi(g', g) = 0 \quad (1.14)$$

which implies associativity of the multiplication law (1.12). Had we chosen different constants  $c'_g = c_g + \delta_g$  in (1.9),  $\xi$  would be redefined according to

$$\xi(g'g) \mapsto \xi'(g', g) = \xi(g', g) - \delta_{g'g} + \delta_{g'} + \delta_g, \quad (1.15)$$

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<sup>3</sup> To ease comparison with similar formulae in the literature we remark that the transformation law (1.10) may alternatively be written in the form  $T_g \psi = (\exp(i \tilde{f}_g) \psi) \circ g^{-1}$ , where  $\tilde{f}_g = f_g \circ g$ . Using expression (1.9) with our choice of  $c_g$ , this leads to  $\tilde{f}_g(\{\vec{x}_a\}, t) = \vec{v} \cdot R \cdot \vec{R} + \frac{1}{2} \vec{v}^2 t$ .

which also satisfies (1.14). But the crucial observation is that the phases  $\xi$  cannot be made to zero by such a redefinition. Indeed, if such a choice existed, it would at the same time remove all phase factors on subgroups. That this is not possible can be easily seen by restricting to the abelian subgroup of translations and boosts on which the induced multiplier phases are  $\xi((\vec{v}', \vec{a}'), (\vec{v}, \vec{a})) = \vec{v}' \cdot \vec{a}$ . But on abelian groups a redefinition of the form (1.15) is necessarily symmetric in the group elements  $g'$  and  $g$  and cannot possibly remove a non-symmetric expression.

Due to the unavoidable presence of the phase factors in the transformation law (1.11) we do not have a representation of the Galilei group on our Hilbert space, but rather a projective representation. Equivalently [9], we can have a proper representation of a slightly larger group, the so-called extended Galilei group  $\bar{\mathcal{G}}$ , which is a central extension of  $\mathcal{G}$  by a group  $\mathcal{Z}$  isomorphic to  $\mathbf{R}^4$ . This means that all of  $\mathcal{Z}$  lies in the centre of  $\bar{\mathcal{G}}$  and that the quotient group  $\bar{\mathcal{G}}/\mathcal{Z}$  is isomorphic to  $\mathcal{G}$ . A group element  $\bar{g} \in \bar{\mathcal{G}}$  is now written in the form  $\bar{g} = (\theta, g)$ , where  $\theta \in \mathcal{Z}$  and  $g \in \mathcal{G}$  as in (1.1). The multiplication law is given by

$$\bar{g}'\bar{g} = (\theta', g')(\theta, g) = (\theta' + \theta + \xi(g', g), g'g) \quad (1.16)$$

which serves to define the following unitary representation  $\bar{U}$  of  $\bar{\mathcal{G}}$  on  $\mathcal{H}$ :

$$\bar{U}_{\bar{g}}\psi = \exp\left(\frac{i}{\hbar}M\theta\right) U_g\psi, \quad (1.17)$$

$$\bar{U}_{\bar{g}'}\bar{U}_{\bar{g}} = \bar{U}_{\bar{g}'\bar{g}}, \quad (1.18)$$

with  $U_g\psi$  as in (1.11).

From (1.11) and (1.17) we infer that an infinitesimal transformation  $\bar{U}_{\delta\bar{g}}$  with infinitesimal parameters  $\delta R_{ab} = \varepsilon_{abc}\delta k_b$ ,  $\delta \vec{v}$ ,  $\delta \vec{a}$ ,  $\delta b$ , and  $\delta \theta$  is given by

$$\bar{U}_{\delta\bar{g}} = \delta \vec{k} \cdot \vec{D} + \delta \vec{v} \cdot \vec{V} + \delta \vec{a} \cdot \vec{A} + \delta b \mathbf{B} + \delta \theta \mathbf{Z}, \quad (1.19)$$

where

$$\vec{D} := - \sum_{i=1}^n (\vec{x}_i \times \vec{\nabla}_i) \quad (1.20a)$$

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<sup>4</sup> As is e.g. shown in [9], ray representations are in bijective correspondence to central  $U(1)$ -extensions. Here we consider  $\mathbf{R}$ -extensions which form the universal cover. This becomes significant in section 2.

$$\vec{V} := \frac{i}{\hbar} M \vec{R} \quad (1.20b)$$

$$\vec{A} := - \sum_{i=1}^n \vec{\nabla}_i \quad (1.20c)$$

$$B := \frac{i}{\hbar} H \quad (1.20d)$$

$$Z := \frac{i}{\hbar} M I. \quad (1.20e)$$

The symbol  $I$  denotes the identity operator. The generators (1.20) satisfy the Lie algebra relations for the group  $\bar{\mathcal{G}}$ , whose non-vanishing commutators are given by

$$[D_a, D_b] = \varepsilon_{abc} D_c \quad (1.21a)$$

$$[D_a, V_b] = \varepsilon_{abc} V_c \quad (1.21b)$$

$$[D_a, A_b] = \varepsilon_{abc} A_c \quad (1.21c)$$

$$[V_a, A_b] = \delta_{ab} Z \quad (1.21d)$$

$$[V_a, B] = A_a. \quad (1.21e)$$

We call this Lie algebra  $\bar{\mathcal{L}}$ . It differs from the corresponding one for the Galilei group only by (1.21d) whose right hand side is now proportional to the central element  $Z$  instead of being zero. Besides  $Z$  there are two more central elements in the enveloping algebra of  $\bar{\mathcal{L}}$  (i.e. Casimir elements):

$$\vec{S}^2 = (Z \vec{D} - \vec{V} \times \vec{A})^2 \quad (1.22)$$

$$K = \vec{A}^2 - 2ZB. \quad (1.23)$$

In the representation (1.20) the vector operator  $\vec{S}$  has the interpretation of  $M/\hbar^2$  times the internal angular momentum<sup>5</sup> and  $K$  corresponds to  $2M/\hbar^2$  times the internal energy, i.e. the total energy minus the kinetic energy of the center of mass motion. This interpretation may be taken over to any irreducible representation corresponding to strictly positive eigenvalues of the Casimir element  $Z$ . In view of (1.20e),  $Z$  is sometimes given the interpretation of  $\frac{i}{\hbar} M$  with  $M$  as operator for the overall mass. But since mass is none of our dynamical variables this does, in our opinion, not really make much sense in the present context.

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<sup>5</sup> The term ‘spin’ is already reserved for the possible internal angular momentum of each particle. We shall not use this term for the translation-invariant part of angular momentum.

The superselection rule first stated by Bargmann [1] is usually motivated in the following manner: Let us restrict to the subgroup generated by space translations and boosts and let  $g = (1, 0, \vec{a}, 0)$  and  $g' = (1, \vec{v}, 0, 0)$  be the group elements for a spatial translation and a boost respectively. On one hand we have  $g'^{-1}g^{-1}g'g = 1$ , whereas the corresponding operators on the Hilbert space,  $U_{g'}$  and  $U_g$ , obey  $U_{g'}^{-1}U_g^{-1}U_{g'}U_g = \exp(-\frac{i}{\hbar}M\vec{v}' \cdot \vec{a})I$  (compare (1.12 – 13)). Applying this combination to a superposition  $\psi_M + \psi_{M'}$  results in a relative phase factor of  $\exp(\frac{i}{\hbar}\vec{v}' \cdot \vec{a}(M - M'))$  and therefore a different ray, unless  $M = M'$ . But the identity Galilei transformation should not alter physical states. Hence  $\psi_M + \psi_{M'}$  cannot represent a physical state if  $M \neq M'$ . Let us describe this situation in a slightly more geometric fashion. We are given two Hilbert spaces,  $\mathcal{H}_M$  and  $\mathcal{H}_{M'}$ , and their associated projective spaces of rays,  $\mathcal{PH}_M$  and  $\mathcal{PH}_{M'}$ . That each Hilbert space carries a projective representation of  $\mathcal{G}$  means that  $\mathcal{G}$  acts on  $\mathcal{PH}_M$  and  $\mathcal{PH}_{M'}$ . But, as just shown,  $\mathcal{G}$  does not act on  $\mathcal{P}(\mathcal{H}_M \oplus \mathcal{H}_{M'})$ . The largest subset it acts on is the disjoint union  $\mathcal{PH}_M \cup \mathcal{PH}_{M'}$ . Elements in the complement cannot belong to the set of pure states which, by assumption, admits an action of the Galilei group.

Let us now look at this situation from the extended Galilei group  $\bar{\mathcal{G}}$ . The left hand side of (2.21d) just represents the infinitesimal version of the combination of translations and boosts given above. The requirement that physical states are unchanged by this combination is then equivalent to the requirement that physical states are eigenstates<sup>6</sup> of  $M = -i\hbar Z$ . Geometrically the situation is now different. We now have proper actions of  $\bar{\mathcal{G}}$  on all the Hilbert spaces and hence on all corresponding projective spaces of rays. However, whereas  $Z$  acts trivially on  $\mathcal{PH}_M$  and  $\mathcal{PH}_{M'}$ , it acts non-trivially on  $\mathcal{P}(\mathcal{H}_M \oplus \mathcal{H}_{M'})$ . The superselection principle now declares that pure states must be fixed-points of  $Z$ 's action, which for  $\mathcal{P}(\mathcal{H}_M \oplus \mathcal{H}_{M'})$  are given by the set  $\mathcal{PH}_M \cup \mathcal{PH}_{M'}$ .

We have argued at the beginning that in order to make sense of a mass superselection rule one should regard mass as dynamical variable. In the quantum theory, its associated self-adjoint operator,  $M$ , then generates a one-parameter group of unitary transformations which we may identify with  $Z$ . Any theory with dynamical mass is therefore expected to admit  $\bar{\mathcal{G}}$  rather than  $\mathcal{G}$  as symmetry group. In the next section we show how this can arise at the classical level.

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<sup>6</sup> We shall generally refer to generalized eigenstates, which occur for operators with continuous spectra, also simply as eigenstates. The appropriate constructions – well known from standard quantum mechanics – are implicitly understood at these points.

## Section 2

In this section we minimally extend the Hamiltonian system considered so far in order to also treat the masses  $\{m_i\}$  in a dynamical fashion. The idea is really simple, namely to just retain the original Hamiltonian function,

$$H = \sum_{i=1}^n \frac{\vec{p}_i^2}{2m_i} + V(\{\vec{x}_i\}, \{m_i\}), \quad (2.1)$$

and adjoin the  $2n$  real-valued canonical variables  $\{(\zeta_i, m_i)\}$ ,  $i = 1, \dots, n$ . We consider  $\{\zeta_i\}$  as new coordinates and  $\{m_i\}$  the corresponding momenta. They satisfy the standard canonical commutation relations<sup>7</sup>  $\{\zeta_i, m_j\} = \delta_{ij}$ . Correspondingly, we have to add a kinetic term  $\sum_i m_i \dot{\zeta}_i$  to the action:

$$S = \int dt \left\{ \sum_{i=1}^n \left( \vec{p}_i \dot{\vec{x}}_i + m_i \dot{\zeta}_i \right) - \sum_{i=1}^n \frac{\vec{p}_i^2}{2m_i} - V(\{\vec{x}_i\}, \{m_i\}) \right\}. \quad (2.2)$$

Variations with respect to  $\vec{x}_i$ ,  $\vec{p}_i$ ,  $\zeta_i$ , and  $m_i$  yield the Hamiltonian equations:

$$\dot{\vec{p}}_i = -\vec{\nabla}_{x_i} V \quad (2.3a)$$

$$\dot{\vec{x}}_i = \frac{\vec{p}_i}{m_i} \quad (2.3b)$$

$$\dot{m}_i = 0 \quad (2.3c)$$

$$\dot{\zeta}_i = \nabla_{m_i} V - \frac{\vec{p}_i^2}{2m_i^2}. \quad (2.3d)$$

Equations (2.3c) just imply conservation of the individual masses. Inserting constant  $m_i$  into (2.3a – b) yields the standard equations which also allow to derive the law of energy conservation in its familiar form:  $\sum_i \frac{1}{2} m_i \dot{\vec{x}}_i^2 + V = \text{const}$ . Having found solutions  $(\{\vec{x}_i(t)\}, \{\vec{p}_i(t)\})$  we can easily integrate (2.3d):

$$\zeta_i(t) = \int^t dt' \left\{ \nabla_{m_i} V(\{\vec{x}_i(t')\}, \{m_i\}) - \frac{1}{2} \vec{x}_i^2(t') \right\}. \quad (2.4)$$

Next we wish to investigate the invariance properties of the given dynamics under the Galilei group. We assume invariance of the masses  $m_i$  which obviously

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<sup>7</sup> We allow  $m_i$  to take values on the whole real axis. If one restricted  $m_i$  to the positive real axis, the procedure would be different at this point.

implies the invariance of (2.3a–c) under general Galilei transformations. To investigate invariance of (2.3d), we write down this equation for the transformed solution curves, insert  $\vec{p}'_i(t) = R\vec{p}_i(t) + m_i\vec{v}$ , and subtract (2.3d) for the untransformed solution curves. This yields to

$$\begin{aligned}\dot{\zeta}'_i(t) &= \dot{\zeta}_i(t) - \vec{v} \cdot R \cdot \dot{\vec{x}}_i(t) + \frac{1}{2}\vec{v}^2 t \\ \text{or } \zeta'_i(t) &= \zeta_i(t) - \vec{v} \cdot R \cdot \vec{x}_i(t) - \frac{1}{2}\vec{v}^2 t + \gamma_g,\end{aligned}\quad (2.5)$$

where  $\gamma_g$  is some constant. We will set it to zero.

The transformation law (2.5) does not define an action of the Galilei group on phase space. However, it defines an action of the extended group  $\bar{\mathcal{G}}$ . To see this in more detail, we explicitly display the transformation law on configuration space<sup>8</sup> (including the time axis) for the group element  $\bar{g} = (\theta, R, \vec{v}, \vec{a}, b) \in \bar{\mathcal{G}}$ :

$$\bar{g}(\{\vec{x}_i\}, \{\zeta_i\}, t) = \left( \{R\vec{x}_i + \vec{v}t + \vec{a}\}, \{\zeta_i - (\theta + \vec{v} \cdot R \cdot \vec{x}_i + \frac{1}{2}\vec{v}^2 t)\}, t + b \right). \quad (2.6)$$

It is now straightforward to verify that a transformation  $\bar{g} = (\theta, g)$  followed by a transformation  $\bar{g}' = (\theta', g')$  equals a transformation  $\bar{g}'\bar{g} = (\theta' + \theta + \xi(g', g), g'g)$ , where  $\xi(g', g) = \vec{v}' \cdot R'\vec{a} + \frac{1}{2}\vec{v}^2 b$ , as in (1.13). Relation (1.14) then ensures associativity so that (2.6) defines indeed an action of the group  $\bar{\mathcal{G}}$  on configuration- and phase space (including the time axis). The inverse element to  $(\theta, g)$  is also easily calculated:

$$(\theta, R, \vec{v}, \vec{a}, b)^{-1} = \left( -\theta + \vec{v} \cdot \vec{a} - \frac{1}{2}\vec{v}^2 b, R^{-1}, -R^{-1} \cdot \vec{v}, -R^{-1} \cdot (\vec{a} - \vec{v}b), -b \right). \quad (2.7)$$

Hence for its action on configuration space and time axis:

$$\begin{aligned}(\theta, g)^{-1}(\{\vec{x}_i\}, \{\zeta_i\}, t) &= \\ \left( \{R^{-1} \cdot (\vec{x}_i - \vec{v}(t-b) - \vec{a})\}, \{\zeta_i + \theta + \vec{v} \cdot (\vec{x}_i - \vec{a}) - \frac{1}{2}\vec{v}^2(t-b)\}, t-b \right).\end{aligned}\quad (2.8)$$

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<sup>8</sup> Since the transformation law for the new momenta  $\{m_i\}$  is trivial we may restrict attention to the configuration space.

### Section 3

In the last section we have seen how the extended Galilei group  $\bar{\mathcal{G}}$  arises as the symmetry group on the classical level if the masses are treated as dynamical variables. In this section we show that the same group acts as symmetries in the corresponding quantum mechanical theory. There is therefore no need for a superselection rule in this model.

In the quantum mechanical treatment of this model the Hilbert space is now given by  $\mathcal{H}_{\text{ex}} = L^2(R^{4n}, d^{3n}x d^n\zeta)$ , where  $R^{4n}$  is spanned by the coordinates  $\{\vec{x}_i\}$  and  $\{\zeta_i\}$  and the measure is just the product of the Lebesgue measures. Since the Hamiltonian (2.1) does not depend on the coordinates  $\{\zeta_i\}$  it is convenient to first perform a Fourier transform in the  $\zeta_i - m_i$  variables:

$$\Psi(\{\vec{x}_i\}, \{\zeta_i\}, t) = (2\pi\hbar)^{-\frac{n}{2}} \int d^n m \exp\left(\frac{i}{\hbar} \sum_{i=1}^n m_i \zeta_i\right) \Phi(\{\vec{x}_i\}, \{m_i\}, t). \quad (3.1)$$

The Schrödinger equation for  $\Psi$  is then equivalent to

$$i\hbar \partial_t \Phi(\{\vec{x}_i\}, \{m_i\}, t) = \left\{ - \sum_{i=1}^n \frac{\hbar^2}{2m_i} \Delta_i + V(\{\vec{x}_i\}, \{m_i\}) \right\} \Phi(\{\vec{x}_i\}, \{m_i\}, t). \quad (3.2)$$

Using the Fourier isomorphism we may think of  $\mathcal{H}_{\text{ex}}$  as a direct integral of Hilbert spaces  $\mathcal{H}_{\{m_i\}}$  each of which is isomorphic to  $\mathcal{H} = L^2(R^{3n}, d^3x)$ :

$$\mathcal{H}_{\text{ex}} = \int_{\mathbf{R}^n} d^n m \mathcal{H}_{\{m_i\}}, \quad (3.3)$$

and where a vector  $\Phi$  is considered as a map  $\Phi : R^n \rightarrow \mathcal{H}$ ,  $\{m_i\} \mapsto \Phi_{\{m_i\}}$  with  $\Phi_{\{m_i\}}(\{\vec{x}_i\}) = \Phi(\{\vec{x}_i\}, \{m_i\})$ . It is not difficult to show the invariance of the Schrödinger equation under the group  $\bar{\mathcal{G}}$ . Its action on solutions  $\Psi(\{\vec{x}_i\}, \{\zeta_i\}, t)$  is now simply defined by composition:

$$\bar{T}_{\bar{g}} \Psi := \Psi \circ \bar{g}^{-1} \quad (3.4)$$

where the action of  $\bar{g}^{-1}$  on  $(\{\vec{x}_i\}, \{\zeta_i\}, t)$  is given by (2.8). Using (3.1) this is seen to induce transformations  $T_{\bar{g}}^{\{m_i\}}$  for the  $\Phi_{\{m_i\}}$ :

$$\begin{aligned} \bar{T}_{\bar{g}}^{\{m_i\}} \Phi_{\{m_i\}}(\{\vec{x}_i\}, t) &= \exp\left\{ \frac{i}{\hbar} M \left[ \theta + \vec{v} \cdot (\vec{R} - \vec{a}) - \frac{1}{2} \vec{v}^2 (t - b) \right] \right\} \\ &\times \Phi_{\{m_i\}}(g^{-1}(\{\vec{x}_i\}, t)). \end{aligned} \quad (3.5)$$

Comparing this to (1.10) and (1.17) we see that this is just the action of  $\bar{\mathcal{G}}$  on the solution space of the Schrödinger equation found in section 1. For each set  $\{m_i\}$  we now have a representation,  $\bar{U}^{\{m_i\}} : g \rightarrow U^{\{m_i\}}$ , given by (compare (1.11)):

$$\begin{aligned}\bar{U}_g^{\{m_i\}} \Phi_{\{m_i\}}(\{\vec{x}_i\}) &= \exp \left\{ \frac{i}{\hbar} M \left[ (\vec{v} \cdot (\vec{R} - \vec{a}) + \frac{1}{2} \vec{v}^2 b) \right] \right\} \\ &\quad \times \left( \exp \left( \frac{i}{\hbar} H b \right) \Phi_{\{m_i\}} \right) (\{R^{-1}(\vec{x}_i - \vec{a} + \vec{v}b)\})\end{aligned}\quad (3.6)$$

where  $M$ ,  $\vec{R}$ , and  $H$  should be understood as functions of the mass variables  $\{m_i\}$ . Hence, the representation of  $\bar{\mathcal{G}}$  on  $\mathcal{H}_{\text{ex}}$  can be written as a direct integral of representations

$$\bar{U} = \int_{\mathbf{R}^n} d^n m \bar{U}^{\{m_i\}}. \quad (3.7)$$

The representation  $\bar{U}^{\{m_i\}}$ , restricted to the central subgroup  $\mathcal{Z}$  (isomorphic to  $\mathbf{R}$  and generated by  $Z$ ) has a kernel which is given by  $\{\frac{\hbar}{M} 2\pi q, q \in \mathbf{Z}\}$ , where  $\mathbf{Z}$  denotes the integers. This defines a different subgroup of  $\mathcal{Z}$  for different  $M$ , thereby showing again that representations for different  $M$  are inequivalent. The requirement on observables to commute with the action of  $\mathcal{Z}$ , which forms the group of translations along the “diagonal” in  $\zeta$ -space, then gives rise to the continuous superselection rule<sup>9</sup> for the overall mass. But this superselection rule is by no means necessary in order to implement an action of the classical symmetry group on quantum mechanical state space. The price to pay is to recognize  $\mathcal{G}$  rather than  $\mathcal{G}$  as classical symmetry group.

## Discussion

The transformation (3.1) should be considered as expansion in common eigenstates of the operators  $-i\hbar\nabla_{\zeta_i}$  which generate translations in  $\zeta_i$  and correspond to the operators for the individual mass  $m_i$ . The Schrödinger equation allows to separate the  $\zeta_i$  motions, just like the center of mass motion is separable in standard translation invariant problems. Expanding in plane waves for these ignorable coordinates leads to a reduced equation which in our case is just the ordinary Schrödinger equation for fixed masses. It is indeed instructive to compare the situation to ordinary quantum mechanics in a translation invariant context. In the latter case,

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<sup>9</sup> For some background material on continuous superselection rules, see e.g. [10][11].

translation invariance is not interpreted to generally prevent us from forming superpositions of plane waves which correspond to quasi localized wave packets. Clearly, in order to prepare such states we need to break translation invariance. The resulting states are then not momentum eigenstates and to manufacture them we need operators which do not commute with translations. We usually do not regard this as a difficulty. Quite the contrary, in order to view translation invariance as a proper physical symmetry we have to regard the translated states as equally valid but decidedly different states. This is what distinguishes a symmetry from a mere redundancy. Redundancies are described by gauge symmetries which are conceptually different from physical symmetries and also lead to different mathematical consequences. Regarding translations as gauge symmetries is equivalent to saying that motions in the translational directions do not change physical states. But in our example, translating the  $\zeta_i$ 's means to change the physical state. For example, shifting the system in real time along the  $\zeta_i$ 's costs action, according to (2.2). This would not be the case if we were considering pure gauge degrees of freedom. Within the framework of our dynamical model we thus talk about physically existing degrees of freedom. Stating a superselection rules for the masses must therefore be equivalent to stating that for some *physical* reason we cannot localize the system in  $\zeta_i$ -space. It seems plausible that many derivations of superselection rules from purely formal arguments in fact make at least one contingent physical assumption of that sort. For better understanding the actual physical input one should in our opinion 1.) find the right dynamical theory in which the relevant quantities are manifestly dynamical and 2.) address the question of what is actually measurable within that framework. Similar views were also expressed in [12]. The present model with dynamical masses is also meant to illustrate this point of view.

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